ON UNIPOTENT AND NILPOTENT PIECES FOR CLASSICAL GROUPS

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Abstract. We show that the definition of unipotent (resp. nilpotent) pieces for classical groups given by Lusztig (resp. Lusztig and the author) coincides with the combinatorial definition using closure relations on unipotent (resp. nilpotent) classes. Moreover we give a closed formula for a map from the set of unipotent (resp. nilpotent) classes in characteristic 2 to the set of unipotent classes in characteristic 0 such that the fibers are the unipotent (resp. nilpotent) pieces.

1. Introduction

Let G be a symplectic or special orthogonal group defined over an algebraically closed field of characteristic exponent $p \geq 1$ and let $\mathfrak{g}$ be the Lie algebra of G. Denote $\mathcal{U}_G$ (resp. $\mathcal{N}_{\mathfrak{g}}$) the set of unipotent (resp. nilpotent) elements in G (resp. $\mathfrak{g}$). In [5, 6], Lusztig defines a partition of $\mathcal{U}_G$ into smooth locally closed G-stable pieces, called unipotent pieces (see [5] for symplectic groups and [6] for special orthogonal groups). In [7], Lusztig proposes another way to define unipotent pieces and shows that the new definition unifies the definitions in [5, 6]. In Appendix A of [7], Lusztig and the author define an analogue partition of $\mathcal{N}_{\mathfrak{g}}$ into smooth locally closed G-stable pieces, called nilpotent pieces. The unipotent or nilpotent pieces are indexed by unipotent classes in the group over $\mathbb{C}$ of the same type as G, and in many ways depend very smoothly on p.

In section 4, we show that one can define pieces combinatorially using closure relations on classes (for unipotent pieces this definition is first considered by Spaltenstein [11]) and that the combinatorial definition gives rise to the same pieces as in [7] (for unipotent pieces in symplectic groups this follows from [5, 7]). In section 5, we determine which classes lie in the same piece (for pieces in symplectic groups this follows from [5, 7]; for unipotent pieces in special orthogonal groups another computation using different methods is given in [9]). In section 6, we define a partition of $\mathcal{U}_G$ (resp. $\mathcal{N}_{\mathfrak{g}}$) into special pieces as in [4] (where $p = 1$) and show that a special piece is a union of unipotent (resp. nilpotent) pieces (for $\mathcal{U}_G$ this follows implicitly from [11, III], see [9]).

2. Notations and recollections

2.1. Let $\mathcal{P}(n)$ denote the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ such that $|\lambda| := \sum \lambda_i = n$. For $\lambda \in \mathcal{P}(n)$, define $\lambda^*_j = |\{\lambda_i | \lambda_i \geq j\}|$ and $m_\lambda(j) = \lambda^*_j - \lambda^*_{j+1}$. For $\lambda, \mu \in \mathcal{P}(n)$, we say that $\lambda \leq \mu$ if the following equivalent conditions hold

(a) $\sum_{j\in[1,i]} \lambda_j \leq \sum_{j\in[1,i]} \mu_j$, for all $i \geq 1$,

(a′) $\sum_{j\in[1,i]} \lambda^*_j \geq \sum_{j\in[1,i]} \mu^*_j$, for all $i \geq 1$.

Let $\mathcal{P}_2(n)$ denote the set of all pairs of partitions (α,β) such that $|\alpha|+|\beta|=n$. For $(\alpha,\beta)\in\mathcal{P}_2(n)$, $\alpha=(\alpha_1\geq\alpha_2\geq\cdots)$, $\beta=(\beta_1\geq\beta_2\geq\cdots)$, we set

$$A_i=\sum_{j\in[1,i]}(\alpha_j+\beta_j),\quad B_i=\sum_{j\in[1,i-1]}(\alpha_j+\beta_j)+\alpha_i,\ i\geq 1. \tag{1}$$

For $(\alpha,\beta),(\alpha',\beta')\in\mathcal{P}_2(n)$, we say that $(\alpha,\beta)\leq(\alpha',\beta')$ if $A_i\leq A_i',\ B_i\leq B_i',\ \text{ for all } i\geq 1$.

2.2. Let $\mathbf{W}$ be the Weyl group of G and $\mathbf{W}^\wedge$ the set of irreducible characters of $\mathbf{W}$ over $\mathbb{C}$.

If $\mathbf{W}$ is of type B_n (or C_n), $n\geq 1$, then $\mathbf{W}^\wedge$ is parametrized by ordered pairs of partitions $(\alpha,\beta)\in\mathcal{P}_2(n)$ (see [2]). We identify $\mathbf{W}^\wedge$ with $\mathcal{P}_2(n)$ where $(n,-)$ is the trivial character and $(-,1^n)$ is the sign character.

If $\mathbf{W}$ is of type D_n, $n\geq 2$, then $\mathbf{W}^\wedge$ is parametrized by unordered pairs of partitions $\{\alpha,\beta\}$ with $|\alpha|+|\beta|=n$, where each pair $\{\alpha,\alpha\}$ corresponds to two (degenerate) elements of $\mathbf{W}^\wedge$. We identify $\mathbf{W}^\wedge$ with the set $\{(\alpha,\beta)\in\mathcal{P}_2(n)|\beta_1\leq\alpha_1\}$, where each pair (α,α) is counted twice.

2.3. Denote Ω_G^p and $\Omega_{\mathfrak{g}}^p$ the set of unipotent classes in $\mathcal{U}_G$ and nilpotent classes in $\mathcal{N}_{\mathfrak{g}}$ respectively. Recall that we have injective maps (see [13, 3, 14])

$$\gamma_G^p:\Omega_G^p\to\mathbf{W}^\wedge,\ \gamma_{\mathfrak{g}}^p:\Omega_{\mathfrak{g}}^p\to\mathbf{W}^\wedge \tag{2}$$

which map a class c to the irreducible character of $\mathbf{W}$ corresponding to the pair $(\mathrm{c},1)$ under Springer correspondence. We denote Λ_G^p (resp. $\Lambda_{\mathfrak{g}}^p$) the image of the map γ_G^p (resp. $\gamma_{\mathfrak{g}}^p$). We may write Ω^p to denote either Ω_G^p or $\Omega_{\mathfrak{g}}^p$ and similar conventions apply for Λ^p, γ^p.

When $p\neq 2$, there are natural identifications of the sets $\Omega_G^p,\Omega_{\mathfrak{g}}^p$ with Ω_G^1, the sets Λ_G^p, $\Lambda_{\mathfrak{g}}^p$ with Λ_G^1, and the maps γ_G^p, $\gamma_{\mathfrak{g}}^p$ with γ_G^1. We have (see [2, 10, 12, 15])

$$\begin{aligned}
&\Lambda^1_{SO(2n+1)}=\{(\alpha,\beta)\in\mathcal{P}_2(n)|\alpha_{i+1}\leq\beta_i\leq\alpha_i+2\},\\
&\Lambda^1_{Sp(2n)}=\{(\alpha,\beta)\in\mathcal{P}_2(n)|\alpha_{i+1}-1\leq\beta_i\leq\alpha_i+1\},\\
&\Lambda^1_{SO(2n)}=\{(\alpha,\beta)\in\mathcal{P}_2(n)|\alpha_{i+1}-2\leq\beta_i\leq\alpha_i\};\\
&\Lambda^2_{SO(2n+1)}=\Lambda^2_{Sp(2n)}=\{(\alpha,\beta)\in\mathcal{P}_2(n)|\alpha_{i+1}-2\leq\beta_i\leq\alpha_i+2\},\\
&\Lambda^2_{SO(2n)}=\{(\alpha,\beta)\in\mathcal{P}_2(n)|\alpha_{i+1}-4\leq\beta_i\leq\alpha_i\};\\
&\Lambda^2_{\mathfrak{so}(2n+1)}=\{(\alpha,\beta)\in\mathcal{P}_2(n)|\beta_i\leq\alpha_i+2\},\ \Lambda^2_{\mathfrak{sp}(2n)}=\mathcal{P}_2(n),\\
&\Lambda^2_{\mathfrak{so}(2n)}=\{(\alpha,\beta)\in\mathcal{P}_2(n)|\beta_i\leq\alpha_i\},
\end{aligned}$$

where for $G=SO(2n)$, each pair (α,α) is counted twice in the sets Λ. Note that (see also [8])

$$\Lambda_G^1\subset\Lambda_G^2\subset\Lambda_{\mathfrak{g}}^2.$$

2.4. Assume $p\neq 2$ and $G=Sp(N)$ (resp. $SO(N)$). The $Sp(N)$ (resp. $O(N)$)-conjugacy class of $u\in\mathcal{U}_G$ is characterized by the partition $\lambda\in\mathcal{P}(N)$ given by the sizes of Jordan blocks of $u-1$. We can identify

$$\begin{aligned}
&\Omega^1_{Sp(N)} \text{ with the set } \{\lambda\in\mathcal{P}(N)|\ m_\lambda(i) \text{ is even if } i \text{ is odd}\},\\
&\Omega^1_{SO(N)} \text{ with the set } \{\lambda\in\mathcal{P}(N)|\ m_\lambda(i) \text{ is even if } i\neq 0 \text{ is even}\},
\end{aligned}$$

where in the case of $SO(2n)$, each λ with all parts even corresponds to two (degenerate) classes.

Assme $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \in \Omega^1_G$ and $\gamma^1_G(\lambda) = (\alpha, \beta)$, $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots)$, $\beta = (\beta_1 \geq \beta_2 \geq \cdots)$. Recall that λ and (α, β) are related as follows [2]. If $G = SO(2n+1)$, then

$$\lambda_{2i-1} = 2\alpha_i + 1 + \delta_i, \quad \lambda_{2i} = 2\beta_i - 1 + \theta_i,$$

where

$$\delta_i = \begin{cases} 1 & \text{if } \beta_i = \alpha_i + 2 \\ -1 & \text{if } \alpha_i = \beta_{i-1}\ (i \geq 2) \\ 0 & \text{otherwise} \end{cases}, \quad \theta_i = \begin{cases} 1 & \text{if } \beta_i = \alpha_{i+1} \\ -1 & \text{if } \beta_i = \alpha_i + 2 \\ 0 & \text{otherwise} \end{cases}; \tag{3}$$

if $G = Sp(2n)$, then

$$\lambda_{2i-1} = 2\alpha_i + \delta_i, \quad \lambda_{2i} = 2\beta_i + \theta_i,$$

where

$$\delta_i = \begin{cases} 1 & \text{if } \beta_i = \alpha_i + 1 \\ -1 & \text{if } \alpha_i = \beta_{i-1} + 1\ (i \geq 2) \\ 0 & \text{otherwise} \end{cases}, \quad \theta_i = \begin{cases} 1 & \text{if } \beta_i = \alpha_{i+1} - 1 \\ -1 & \text{if } \beta_i = \alpha_i + 1 \\ 0 & \text{otherwise} \end{cases};$$

if $G = SO(2n)$, then

$$\lambda_{2i-1} = 2\alpha_i - 1 + \delta_i, \quad \lambda_{2i} = 2\beta_i + 1 + \theta_i,$$

where

$$\delta_i = \begin{cases} 1 & \text{if } \beta_i = \alpha_i \\ -1 & \text{if } \alpha_i = \beta_{i-1} + 2\ (i \geq 2) \\ 0 & \text{otherwise} \end{cases}, \quad \theta_i = \begin{cases} 1 & \text{if } \beta_i = \alpha_{i+1} - 2 \\ -1 & \text{if } \beta_i = \alpha_i \\ 0 & \text{otherwise} \end{cases}$$

(note that $\gamma^1_{SO(2n)}(\lambda) = (\alpha, \alpha)$ if and only if all λ_i are even; the two degenerate classes corresponding to λ are mapped to the two degenerate elements of $\mathbf{W}^\wedge$ corresponding to (α, α) respectively under $\gamma^1_{SO(2n)}$).

2.5. Assume $p = 2$ and $G = Sp(2n)$ (resp. $SO(2n)$). The $Sp(2n)$ (resp. $O(2n)$)-conjugacy class of $u \in \mathcal{U}_G$ is characterized by the partition $\lambda \in \mathcal{P}(2n)$ given by the sizes of Jordan blocks of $u - 1$ and a map $\varepsilon : \mathbb{N} \to \{\omega, 0, 1\}$ satisfying the following conditions (see [11, I 2.6])

(a) $\varepsilon(i) = \omega$, if i is odd, or if $i \geq 1$ and $m_\lambda(i) = 0$,
(b) $\varepsilon(i) = 1$, if $i \neq 0$ is even and $m_\lambda(i)$ is odd,
(c) $\varepsilon(i) \neq \omega$, if i is even and $m_\lambda(i) > 0$.
(d) $\varepsilon(0) = 1$ (resp. $\varepsilon(0) = 0$).

If $G = SO(2n)$, then λ^*_1 is even. We have a natural bijection

$$\Omega^2_{Sp(2n)} \to \Omega^2_{SO(2n+1)}, \ (\lambda, \varepsilon) \mapsto (\lambda', \varepsilon') \tag{4}$$

given by the special isogeny $SO(2n+1) \to Sp(2n)$, where $\lambda = (\lambda_i), \lambda' = (\lambda'_i)$; if $\lambda_{i-1} > 0$ and $\lambda_i = 0$, then $\lambda'_i = 1$, $\varepsilon'(\lambda'_i) = \omega$; otherwise, $\lambda'_i = \lambda_i$, $\varepsilon'(\lambda'_i) = \varepsilon(\lambda_i)$. We identify Ω^2_G with the set of all (λ, ε) as above, where in the case of $SO(2n)$ there are two (degenerate) classes corresponding to each (λ, ε) such that $\varepsilon(\lambda_i) = 0$ for all λ_i with $m_\lambda(\lambda_i) > 0$.

Assume $(\lambda, \varepsilon) \in \Omega^2_G$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, and $\gamma^2_G((\lambda, \varepsilon)) = (\alpha, \beta)$, $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots)$, $\beta = (\beta_1 \geq \beta_2 \geq \cdots)$. Recall that (λ, ε) and (α, β) are related as follows [2]. If $G = Sp(2n)$, then

$$\lambda_{2i-1} = 2\alpha_i + \delta_i, \quad \lambda_{2i} = 2\beta_i + \theta_i, \quad \varepsilon(\lambda_{2i-1}) = \varepsilon(\delta_i), \quad \varepsilon(\lambda_{2i}) = \varepsilon(\theta_i), \tag{5}$$

where

$$\delta_i = \begin{cases} 2 & \text{if } \beta_i = \alpha_i + 2 \\ 1 & \text{if } \beta_i = \alpha_i + 1 \\ -2 & \text{if } \alpha_i = \beta_{i-1} + 2 \ (i \geq 2) \\ -1 & \text{if } \alpha_i = \beta_{i-1} + 1 \ (i \geq 2) \\ 0 & \text{otherwise} \end{cases}, \quad \theta_i = \begin{cases} 2 & \text{if } \beta_i = \alpha_{i+1} - 2 \\ 1 & \text{if } \beta_i = \alpha_{i+1} - 1 \\ -2 & \text{if } \beta_i = \alpha_i + 2 \\ -1 & \text{if } \beta_i = \alpha_i + 1 \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

$$\varepsilon(\delta_i) = \begin{cases} 0 & \text{if } \delta_i = \pm 2 \\ \omega & \text{if } \delta_i = \pm 1 \\ 1 & \text{otherwise} \end{cases}, \quad \varepsilon(\theta_i) = \begin{cases} 0 & \text{if } \theta_i = \pm 2 \\ \omega & \text{if } \theta_i = \pm 1 \\ 1 & \text{otherwise} \end{cases};$$

if $G = SO(2n)$, then

$$\lambda_{2i-1} = 2\alpha_i - 2 + \delta_i \qquad \lambda_{2i} = 2\beta_i + 2 + \theta_i, \quad \varepsilon(\lambda_{2i-1}) = \varepsilon(\delta_i), \quad \varepsilon(\lambda_{2i}) = \varepsilon(\theta_i),$$

where

$$\delta_i = \begin{cases} 2 & \text{if } \beta_i = \alpha_i \\ 1 & \text{if } \beta_i = \alpha_i - 1 \\ -2 & \text{if } \alpha_i = \beta_{i-1} + 4 \ (i \geq 2) \\ -1 & \text{if } \alpha_i = \beta_{i-1} + 3 \ (i \geq 2) \\ 0 & \text{otherwise} \end{cases}, \quad \theta_i = \begin{cases} 2 & \text{if } \beta_i = \alpha_{i+1} - 4 \\ 1 & \text{if } \beta_i = \alpha_{i+1} - 3 \\ -2 & \text{if } \beta_i = \alpha_i \\ -1 & \text{if } \beta_i = \alpha_i - 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\varepsilon(\delta_i) = \begin{cases} 0 & \text{if } \delta_i = \pm 2 \\ \omega & \text{if } \delta_i = \pm 1 \\ 1 & \text{otherwise} \end{cases}, \quad \varepsilon(\theta_i) = \begin{cases} 0 & \text{if } \theta_i = \pm 2 \\ \omega & \text{if } \theta_i = \pm 1 \\ 1 & \text{otherwise} \end{cases}$$

(note that $\gamma^2_{SO(2n)}((\lambda,\varepsilon)) = (\alpha,\alpha)$ if and only if for all i, $\varepsilon(\lambda_i) = 0$; the two degenerate classes corresponding to (λ,ε) are mapped to the two degenerate elements of $\mathbf{W}^\wedge$ corresponding to (α,α) respectively under $\gamma^2_{SO(2n)}$).

2.6. Assume $p = 2$ and $G = Sp(N)$ (resp. $SO(N)$). The $Sp(N)$ (resp. $O(N)$)-class of $x \in \mathcal{N}_{\mathfrak{g}}$ is characterized by the partition $\lambda \in \mathcal{P}(N)$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, given by the sizes of the Jordan blocks of x and a map $\chi : \{\lambda_i\}_{i\geq 1} \to \mathbb{N}$ satisfying the following conditions (see [1])

(a) $0 \leq \chi(\lambda_i) \leq \frac{\lambda_i}{2}$ (resp. $[\frac{\lambda_i+1}{2}] \leq \chi(\lambda_i) \leq \lambda_i$),

(b) $\chi(\lambda_i) \geq \chi(\lambda_{i+1}), \lambda_i - \chi(\lambda_i) \geq \lambda_{i+1} - \chi(\lambda_{i+1})$,

(c) $\chi(\lambda_i) = \frac{\lambda_i}{2}$ (resp. $\chi(\lambda_i) = \lambda_i$), if $m_\lambda(\lambda_i)$ is odd.

If $G = SO(N)$, then $\{\lambda_i \neq 0 | m_\lambda(\lambda_i) \text{ is odd}\} = \{a, a-1\} \cap \mathbb{N}$ for some $a \in \mathbb{N}$. We identify $\Omega^2_{\mathfrak{g}}$ with the set of all (λ,χ) as above, where in the case of $SO(2n)$ there are two (degenerate) classes corresponding to each (λ,χ) with $\chi(\lambda_i) = \lambda_i/2$ for all $i \geq 1$.

Assume $(\lambda,\chi) \in \Omega^2_{\mathfrak{g}}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, and $\gamma^2_{\mathfrak{g}}((\lambda,\chi)) = (\alpha,\beta)$, $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots)$, $\beta = (\beta_1 \geq \beta_2 \geq \cdots)$. Recall that (λ,χ) and (α,β) are related as follows [12, 15]. If $G = Sp(2n)$,

then

$$\lambda_1=\begin{cases}\alpha_1+\beta_1 & \text{if } \alpha_1<\beta_1\\ 2\alpha_1 & \text{if } \alpha_1\geq\beta_1\end{cases},\quad \chi(\lambda_1)=\alpha_1,$$

$$\lambda_{2i}=\begin{cases}\alpha_{i+1}+\beta_i & \text{if } \beta_i<\alpha_{i+1}\\ \alpha_i+\beta_i & \text{if } \beta_i>\alpha_i\\ 2\beta_i & \text{if } \alpha_{i+1}\leq\beta_i\leq\alpha_i\end{cases},\quad \chi(\lambda_{2i})=\begin{cases}\alpha_i & \text{if } \beta_i>\alpha_i\\ \beta_i & \text{if } \beta_i\leq\alpha_i\end{cases},$$

$$\lambda_{2i+1}=\begin{cases}\alpha_{i+1}+\beta_i & \text{if } \alpha_{i+1}>\beta_i\\ \alpha_{i+1}+\beta_{i+1} & \text{if } \alpha_{i+1}<\beta_{i+1}\\ 2\alpha_{i+1} & \text{if } \beta_{i+1}\leq\alpha_{i+1}\leq\beta_i\end{cases},\quad \chi(\lambda_{2i+1})=\begin{cases}\alpha_{i+1} & \text{if } \alpha_{i+1}\leq\beta_i\\ \beta_{i+1} & \text{if } \alpha_{i+1}>\beta_i\end{cases},\ i\geq 1;$$

if $G=SO(2n+1)$, let $k\geq 0$ be the largest integer such that $\beta_k>0$, then

$$\lambda_{2i-1}=\begin{cases}\alpha_i+\beta_i & \text{if } i<k+1\\ \alpha_i+1 & \text{if } i=k+1\\ \alpha_i & \text{if } i>k+1\end{cases}\qquad \chi(\lambda_{2i-1})=\begin{cases}\alpha_i+1 & \text{if } i\leq k+1\\ \alpha_i & \text{if } i>k+1\end{cases}$$

$$\lambda_{2i}=\begin{cases}\alpha_i+\beta_i & \text{if } i<k+1\\ \alpha_i & \text{if } i\geq k+1\end{cases}\qquad \chi(\lambda_{2i})=\begin{cases}\alpha_i+1 & \text{if } i<k+1\\ \alpha_i & \text{if } i\geq k+1\end{cases},\ i\geq 1;$$

if $G=SO(2n)$, then

$$\lambda_{2i-1}=\lambda_{2i}=\alpha_i+\beta_i,\quad \chi(\lambda_{2i-1})=\chi(\lambda_{2i})=\alpha_i,\ i\geq 1$$

(note that $\gamma^2_{\mathfrak{so}(2n)}((\lambda,\chi))=(\alpha,\alpha)$ if and only if for all i, $\chi(\lambda_i)=\lambda_i/2$; the two degenerate classes corresponding to (λ,χ) are mapped to the two degenerate elements of $\mathbf{W}^\wedge$ corresponding to (α,α) respectively under $\gamma^2_{\mathfrak{so}(2n)}$).

2.7. Assume $G=Sp(2n)$. When $p\neq 2$, each piece consists of one class. Let $\mathrm{c}_i=(\lambda_{\mathrm{c}_i},\varepsilon_i)\in\Omega^2_G$ (resp. $\mathrm{c}_i=(\lambda_{\mathrm{c}_i},\chi_i)\in\Omega^2_{\mathfrak{g}}$), $i=1,2$. Recall that

Lemma ([7]). *The classes* c_1 *and* c_2 *lie in the same piece if and only if* $\lambda_{\mathrm{c}_1}=\lambda_{\mathrm{c}_2}$.

2.8. Let V be a vector space of dimension N equipped with a nondegenerate quadratic form Q. The orthogonal group $O(V)=\{g\in GL(V)|Q(gv)=v,\forall\ v\in V\}$. The special orthogonal group $SO(V)$ is the identity component of $O(V)$.

Assume $G=SO(V)$. Let $\mathrm{c}\in\Omega^p_G$ (resp. $\Omega^2_{\mathfrak{g}}$) and $u\in\mathrm{c}$ (resp. $x\in\mathrm{c}$). Let $V_*=(V_{\geq a})$ (with $V_{\geq a+1}\subset V_{\geq a}$) be the canonical Q-filtration of V associated to u (resp. x) (see [7, 2.7(a), A.4(a)]). We define $gr_a(V_*)=V_{\geq a}/V_{\geq a+1}$ and set $f_a=\dim gr_a(V_*)$. Then $f_a\neq 0$ for finitely many a, and the set of numbers $\{f_a\}$ depends only on c and not on the choice of $u\in\mathrm{c}$ (resp. $x\in\mathrm{c}$); we denote this set by Υ_{c}. Let $\mathrm{c}_1,\mathrm{c}_2\in\Omega^p$. If $G=SO(2n)$, we assume that c_1, c_2 are not two degenerate classes conjugate under $O(2n)$. Recall that

Lemma ([7]). *The classes* c_1 *and* c_2 *lie in the same piece if and only if* $\Upsilon_{\mathrm{c}_1}=\Upsilon_{\mathrm{c}_2}$.

Let $T=u-1$ (resp. $T=x$). If $p\neq 2$, then (see [7])

(a) $V_{\geq a}=\sum_{j\geq\max(0,a)}T^j(\ker T^{2j-a+1})$.

Now assume $p=2$. If $T=0$, then $V_{\geq a}=0$ for all $a\geq 1$ and $V_{\geq a}=V$ for all $a\leq 0$. Assume from now on that $T\neq 0$. Let e be the smallest integer such that $T^e=0$ and f the smallest integer such

that $QT^f=0$. Let m be the largest integer such that $gr_m(V_*)\neq 0$. We have (see [7])

$$\begin{aligned}
&m=\max(e-1,2f-2);\\
&V_{\geq -m+1}=\{v\in V|T^{e-1}v=0\} \text{ if } e\geq 2f,\\
&V_{\geq -m+1}=\{v\in V|T^{e-1}v=0, Q(T^{f-1}v)=0\} \text{ if } e=2f-1,\\
&V_{\geq -m+1}=\{v\in V|Q(T^{f-1}v)=0\} \text{ if } e<2f-1.
\end{aligned}$$

Let $V'=V_{\geq -m+1}/V_{\geq m}$. Then Q induces a nondegenerate quadratic form on V' and u (resp. x) induces an element $u'\in\mathcal{U}_{SO(V')}$ (resp. $x'\in\mathcal{N}_{\mathfrak{so}(V')}$), where $SO(V')$ is defined with respect to Q'. Let c' be the class of u' (resp. x') in $SO(V')$ (resp. $\mathfrak{so}(V')$). Assume $\Upsilon_{\mathrm{c}}=\{f_a\}$ and $\Upsilon_{\mathrm{c}'}=\{f'_a\}$. We have (see [7])

(b) $f_a=f'_a$, *for all* $a\in[-m+1,m-1]$.

2.9. Let $\mathrm{c},\mathrm{c}'\in\Omega^p_G$ (resp. $\Omega^p_{\mathfrak{g}}$). We say that $\mathrm{c}\leq\mathrm{c}'$, if c is contained in the closure of c' in G (resp. $\mathfrak{g}$); and that $\mathrm{c}<\mathrm{c}'$, if $\mathrm{c}\leq\mathrm{c}'$ and $\mathrm{c}\neq\mathrm{c}'$.

In the following if $G=SO(2n)$, we assume that c and c' are not two degenerate classes conjugate under $O(2n)$ (such two classes are incomparable with respect to the partial order $\leq$).

Assume $\mathrm{c}=\lambda,\mathrm{c}'=\lambda'\in\Omega^1_G$. We have $\mathrm{c}\leq\mathrm{c}'$ if and only if $\lambda\leq\lambda'$ (see [11, II 8.2]).

Assume $\mathrm{c}=(\lambda,\varepsilon),\mathrm{c}'=(\mu,\phi)\in\Omega^2_G$. We order the set $\{\omega,0,1\}$ by $\omega<0<1$. Then $\mathrm{c}\leq\mathrm{c}'$ if and only if $(\lambda,\varepsilon)\leq(\mu,\phi)$ (see [11, II 8.2]), namely, the following conditions hold

(a) $\lambda\leq\mu$,

(b) $\sum_{j\in[1,i]}\lambda^*_j-\max(\varepsilon(i),0)\geq\sum_{j\in[1,i]}\mu^*_j-\max(\phi(i),0)$, for all $i\geq 1$,

(c) if $\sum_{j\in[1,i]}\lambda^*_j=\sum_{j\in[1,i]}\mu^*_j$ and $\lambda^*_{i+1}-\mu^*_{i+1}$ is odd, then $\phi(i)\neq 0$, for all $i\geq 1$.

3. Reformulation of closure relations on unipotent and nilpotent classes

3.1. Let $\mathrm{c},\mathrm{c}'\in\Omega^p$ (if $G=SO(2n)$, we assume that c, c' are not two degenerate classes conjugate under $O(2n)$).

Proposition. *We have* $\mathrm{c}\leq\mathrm{c}'$ *if and only if* $\gamma^p(\mathrm{c})\leq\gamma^p(\mathrm{c}')$.

If $\mathrm{c},\mathrm{c}'\in\Omega^2_{\mathfrak{g}}$, the proposition is a result of Spaltenstein [12]. The proofs for $\mathrm{c},\mathrm{c}'\in\Omega^p_G$ when $p\neq 2$ and $p=2$ are given in subsections 3.2 and 3.3 respectively.

3.2. Assume $\mathrm{c}=\lambda,\mathrm{c}'=\lambda'\in\Omega^1_G$, $\gamma^1_G(\mathrm{c})=(\alpha,\beta)$ and $\gamma^1_G(\mathrm{c})=(\alpha',\beta')$. We show that

(a) $\lambda\leq\lambda'$ *iff* $(\alpha,\beta)\leq(\alpha',\beta')$.

We prove (a) for $G=SO(2n+1)$. The proofs for $Sp(2n)$ and $SO(2n)$ are entirely similar and omitted. For $(\alpha,\beta)\in\Lambda^1_{SO(2n+1)}$, let A_i,B_i be as in (1) and let $\Delta_i=\sum_{j\in[1,i]}(\delta_j+\theta_j)$, $\Theta_i=\Delta_{i-1}+\delta_i$, where δ_j,θ_j are as in (3). One can easily verify that

$$\Delta_i=\begin{cases}1 & \text{if } \beta_i=\alpha_{i+1}\\ 0 & \text{otherwise}\end{cases},\quad \Theta_i=\begin{cases}1 & \text{if } \beta_i=\alpha_i+2\\ 0 & \text{otherwise}\end{cases}. \tag{7}$$

We have $\sum_{j=1}^{2i}\lambda_j=2A_i+\Delta_i$ and $\sum_{j=1}^{2i-1}\lambda_j=2B_i+\Theta_i+1$. Assume $\lambda\leq\lambda'$. It follows from 2.1 (a) and (7) that $A_i\leq A'_i$ and $B_i\leq B'_i$. Hence $(\alpha,\beta)\leq(\alpha',\beta')$ (see 2.1).

Assume $(\alpha,\beta)\leq(\alpha',\beta')$. Then $A_i\leq A'_i$ and $B_i\leq B'_i$ for all i. We show that $A_i=A'_i$ implies $\Delta_i\leq\Delta'_i$. Assume otherwise, $A_i=A'_i,\Delta_i=1,\Delta'_i=0$. Then $\beta_i=\alpha_{i+1}$ and $\beta'_i>\alpha'_{i+1}$. Since $B_i=A_i-\beta_i\leq B'_i=A'_i-\beta'_i$ and $B_{i+1}=A_i+\alpha_{i+1}\leq B'_{i+1}=A'_i+\alpha'_{i+1}$, we have $\beta_i\geq\beta'_i$ and $\alpha_{i+1}\leq\alpha'_{i+1}$ which is a contradiction. Similarly $B_i=B'_i$ implies $\Theta_i\leq\Theta'_i$. Hence $\lambda\leq\lambda'$.

3.3. Assume $\mathrm{c}=(\lambda,\varepsilon),\mathrm{c}'=(\mu,\phi)\in\Omega^2_G$, $\gamma^2_G(\mathrm{c})=(\alpha,\beta)$ and $\gamma^2_G(\mathrm{c}')=(\alpha',\beta')$. We show that

(a) $(\lambda,\varepsilon)\leq(\mu,\phi)$ *iff* $(\alpha,\beta)\leq(\alpha',\beta')$.

We prove (a) for $G=Sp(2n)$ and then in view of (4), (a) follows for $G=SO(2n+1)$. The proof for $SO(2n)$ is entirely similar and omitted. Since $\sum_{j>i}\lambda^*_j=\sum_{j\in[1,\lambda^*_{i+1}]}(\lambda_j-i)$ and, for i large enough, $\sum_{j\in[1,i]}\lambda^*_j=\sum_{j\in[1,i]}\mu^*_j$, we have

(b) $\sum_{j\in[1,i]}\lambda^*_j=\sum_{j\in[1,i]}\mu^*_j$ *iff* $\sum_{j\in[1,\lambda^*_{i+1}]}(\lambda_j-i)=\sum_{j\in[1,\mu^*_{i+1}]}(\mu_j-i)$.

We show that

(c) *if* $\lambda\leq\mu$ *and* $\sum_{j\in[1,k]}\lambda^*_j=\sum_{j\in[1,k]}\mu^*_j$, *then* $\sum_{j\in[1,\lambda^*_{k+1}]}\lambda_j=\sum_{j\in[1,\lambda^*_{k+1}]}\mu_j$,

(d) *if* $\lambda\leq\mu$ *and* $\sum_{j\in[1,m]}\lambda_j=\sum_{j\in[1,m]}\mu_j$, *then* $\sum_{j\in[1,\mu_m]}\lambda^*_j=\sum_{j\in[1,\mu_m]}\mu^*_j$.

By 2.1 (a′), the assumptions in (c) imply that $\lambda^*_k\leq\mu^*_k$, $\lambda^*_{k+1}\geq\mu^*_{k+1}$. It follows that $\mu_j=k$ for $j\in[\mu^*_{k+1}+1,\lambda^*_{k+1}]$ and thus $\sum_{j\in[1,\mu^*_{k+1}]}(\mu_j-k)=\sum_{j\in[1,\lambda^*_{k+1}]}(\mu_j-k)$. Now (c) follows from (b). By 2.1 (a), the assumptions in (d) imply that $\lambda_{m+1}\leq\mu_{m+1}$ and $\lambda_m\geq\mu_m$. Let $k=\mu_m$. Then $\sum_{j\in[1,\lambda^*_k]}(\lambda_j-k)=\sum_{j\in[1,m]}(\lambda_j-k)$, $\sum_{j\in[1,\mu^*_k]}(\mu_j-k)=\sum_{j\in[1,m]}(\mu_j-k)$ (since $\lambda_i=k$, $i\in[m+1,\lambda^*_k]$; $\mu_i=k$, $i\in[m+1,\mu^*_k]$). Now (d) follows from (b).

For $(\alpha,\beta)\in\Lambda^2_{Sp(2n)}$, let A_i,B_i be as in (1) and let $\Delta_i=\sum_{j\in[1,i]}(\delta_j+\theta_j)$, $\Theta_i=\Delta_{i-1}+\delta_i$, where δ_j and θ_j are as in (6). One can easily verify that we have

$$\Delta_i=\begin{cases}2 & \text{if } \beta_i=\alpha_{i+1}-2\\ 1 & \text{if } \beta_i=\alpha_{i+1}-1\\ 0 & \text{otherwise}\end{cases},\quad \Theta_i=\begin{cases}2 & \text{if } \beta_i=\alpha_i+2\\ 1 & \text{if } \beta_i=\alpha_i+1\\ 0 & \text{otherwise}\end{cases}. \tag{8}$$

Using (5) one can easily check that $\lambda_{2i-1}=\lambda_{2i}$ iff $\beta_i\geq\alpha_i$, or (if $i\geq2$) $\beta_i=\beta_{i-1}$, $\alpha_i=\alpha_{i+1}$ and $\beta_i\leq\alpha_{i+1}-1$; and $\lambda_{2i}=\lambda_{2i+1}$ iff $\beta_i\leq\alpha_{i+1}$, or $\beta_i=\beta_{i+1}$, $\alpha_i=\alpha_{i+1}$ and $\beta_i\geq\alpha_i+1$. It then follows that

(e) *if* $\beta_i=\alpha_i+2$, *then* $\lambda^*_{\lambda_{2i}+1}$ *is even,*

(f) *if* $\alpha_i=\beta_{i-1}+2$ *(*$i\geq2$*), then* $\lambda^*_{\lambda_{2i-1}+1}$ *is odd.*

We have $\sum_{j\in[1,2i]}\lambda_j=2A_i+\Delta_i$ and $\sum_{j\in[1,2i-1]}\lambda_j=2B_i+\Theta_i$.

Assume $(\lambda,\varepsilon)\leq(\mu,\phi)$. It follows from $\lambda\leq\mu$ and (8) that $A_i\leq A'_i$ except if $\Delta_i=0,\Delta'_i=2$ and $\sum_{j\in[1,2i]}\lambda_j=\sum_{j\in[1,2i]}\mu_j$. In the latter case, we have $\beta_i\geq\alpha_{i+1}$, $\beta'_i=\alpha'_{i+1}-2$, $\lambda_{2i+1}\leq\mu_{2i+1}$ and $\lambda_{2i}\geq\mu_{2i}$. Then $\mu_{2i}=\mu_{2i+1}$ and $\phi(\mu_{2i})=0$ (we use (5)). Let $k=\mu_{2i}$. By (d), we have $\sum_{j\in[1,k]}\lambda^*_j=\sum_{j\in[1,k]}\mu^*_j$. By (f), μ^*_{k+1} is odd. If $\lambda_{2i}>k$, then $\lambda^*_{k+1}=2i$ is even, which contradicts 2.9 (c). Hence $\lambda_{2i}=k$ and thus $\varepsilon(k)=0$ (we use 2.9 (b) and k even). It follows that $\beta_i=\alpha_i+2$

(since $\beta_i \geq \alpha_{i+1}$) and thus λ^*_{k+1} is even by (e), which again contradicts 2.9 (c). Hence $A_i \leq A'_i$. Similarly we have $B_i \leq B'_i$. Hence $(\alpha, \beta) \leq (\alpha', \beta')$.

Assume $(\alpha, \beta) \leq (\alpha', \beta')$. We show that if $A_i = A'_i$ then $\Delta_i \leq \Delta'_i$. Assume otherwise, $A_i = A'_i$, $\Delta_i = 1$ (resp. 2), $\Delta'_i = 0$ (resp. 1 or 0). Then as in the proof of 3.2 (a), we have $\beta_i \geq \beta'_i$ and $\alpha_{i+1} \leq \alpha'_{i+1}$, which contradicts to $\Delta_i \leq \Delta'_i$ (we use (8)). Similarly one can show if $B_i = B'_i$ then $\Theta_i \leq \Theta'_i$. It follows that $\lambda \leq \mu$.

We verify 2.9 (b). Assume $\varepsilon(k) = 1$, $\phi(k) \leq 0$, and $\sum_{j\in[1,k]} \lambda^*_j = \sum_{j\in[1,k]} \mu^*_j$. Let $m = \lambda^*_{k+1}$. Then $\lambda_{m+1} = \mu_{m+1} = k$ (since $\mu^*_{k+1} \leq \lambda^*_{k+1} < \lambda^*_k \leq \mu^*_k$). By (c), we have $\sum_{j\in[1,m]} \lambda_j = \sum_{j\in[1,m]} \mu_j$. Suppose $m = 2i$. Note $\varepsilon(\lambda_{m+1}) = 1$ implies that $\delta_{i+1} = 0$, $\Delta_i = 0$ and thus $\lambda_{m+1} = 2\alpha_{i+1}$. Since $A_i \leq A'_i$ and $2A_i + \Delta_i = 2A'_i + \Delta'_i$, we have $A_i = A'_i$ and $\Delta'_i = 0$. Together with $\phi(\mu_{m+1}) \leq 0$, this implies that $\mu_{m+1} > 2\alpha'_{i+1}$. Hence $\alpha_{i+1} > \alpha'_{i+1}$, which contradicts $B_{i+1} \leq B'_{i+1}$. Suppose $m = 2i-1$. Note $\varepsilon(\lambda_{m+1}) = 1$ implies that $\theta_i = 0$, $\Theta_i = 0$ and thus $\lambda_{m+1} = 2\beta_i$. Since $B_i \leq B'_i$ and $2B_i + \Theta_i = 2B'_i + \Theta'_i$, we have $B_i = B'_i$ and $\Theta'_i = 0$. Together with $\phi(\mu_{m+1}) \leq 0$ this implies that $\mu_{m+1} > 2\beta'_i$. Hence $\beta_i > \beta'_i$, which contradicts $A_i \leq A'_i$.

It remains to verify 2.9 (c). Assume $\sum_{j\in[1,k]} \lambda^*_j = \sum_{j\in[1,k]} \mu^*_j$, $\lambda^*_{k+1} - \mu^*_{k+1}$ is odd, and $\phi(k) = 0$. Let $\lambda^*_{k+1} = m$. Then $\mu_m = k$ (since $\mu^*_{k+1} < \lambda^*_{k+1} \leq \lambda^*_k \leq \mu^*_k$). By (c), $\sum_{j\in[1,m]} \lambda_j = \sum_{j\in[1,m]} \mu_j$. Suppose $m = 2i$. Note that $\varepsilon(\mu_m) = 0$ implies that $\theta'_i = 2, \Delta'_i = 2$ ($\beta'_i = \alpha'_{i+1} - 2$) or $\theta'_i = -2, \Delta'_i = 0$ ($\beta'_i = \alpha'_i + 2$). If $\beta'_i = \alpha'_{i+1} - 2$, then $\Delta'_i = 2$ and $2A_i + \Delta_i = 2A'_i + \Delta'_i$ imply that $A_i = A'_i$, $\Delta_i = 2$ and thus $\beta_i = \alpha_{i+1} - 2$, $\theta_i = 2$. Since $\lambda_m = 2\beta_i + \theta_i > \mu_m = 2\beta'_i + \theta'_i$, we have $\beta_i > \beta'_i$ and thus $\alpha_{i+1} > \alpha'_{i+1}$, which contradicts $B_{i+1} \leq B'_{i+1}$. If $\beta'_i = \alpha'_i + 2$, then μ^*_{k+1} is even (see (e)), which contradicts the fact that $\lambda^*_{k+1} - \mu^*_{k+1}$ is odd. Suppose $m = 2i-1$. Note that $\phi(\mu_m) = 0$ implies that $\delta'_i = 2$, $\Theta'_i = 2$ ($\beta'_i = \alpha'_i + 2$) or $\delta'_i = -2$, $\Theta'_i = 0$ ($\alpha'_i = \beta'_{i-1} + 2$). If $\beta'_i = \alpha'_i + 2$, then $\Theta'_i = 2$ and $2B_i + \Theta_i = 2B'_i + \Theta'_i$ imply $\Theta_i = 2$, $B_i = B'_i$ and thus $\beta_i = \alpha_i + 2$, $\delta_i = 2$. Since $\lambda_m = 2\alpha_i + \delta_i > \mu_m = 2\alpha'_i + \delta'_i$, we have $\alpha_i > \alpha'_i$ and thus $\beta_i > \beta'_i$, which contradicts $A_i \leq A'_i$. If $\alpha'_i = \beta'_{i-1} + 2$, then μ^*_{k+1} is odd, which contradicts the fact that $\lambda^*_{k+1} - \mu^*_{k+1}$ is odd. This completes the proof of (a).

4. Combinatorial definition of unipotent and nilpotent pieces

4.1. Let $\tilde{c} \in \Omega^1_G$ and let $c \in \Omega^p$ be such that $\gamma^p(c) = \gamma^1_G(\tilde{c})$. Define $\Sigma^p_{\tilde{c}}$ to be the set of all classes $c' \in \Omega^p$ such that $c' \leq c$ and $c' \nleq c''$ for any $c'' < c$ with $\gamma^p(c'') \in \Lambda^1_G$. We show that

(a) $\{\Sigma^p_{\tilde{c}}\}_{\tilde{c}\in\Omega^1_G}$ *form a partition of* $\mathcal{U}_G$ *or* $\mathcal{N}_{\mathfrak{g}}$ *(see 4.2 (a))*

and that

(b) *each set* $\Sigma^p_{\tilde{c}}$ *is a piece defined in* [7] *(see 5.1 (b)).*

The definition of unipotent pieces using closure relations is first considered by Spaltenstein and (a) for $\mathcal{U}_G$ is shown in [11]. For completeness, we include here a different proof which applies for both unipotent and nilpotent pieces.

We define a map

$$\Phi : \Lambda^p \to \Lambda^1_G,\ (\alpha, \beta) \mapsto (\tilde{\alpha}, \tilde{\beta})$$

as follows. When $p \neq 2$, Φ is the identity map.

For $(\alpha,\beta)\in\Lambda^2_{B_n}$, define $\tilde\alpha_1=\alpha_1$ and

$$\tilde\alpha_i=\begin{cases}[\frac{\alpha_i+\beta_{i-1}}{2}] & \text{if } \alpha_i>\beta_{i-1}\\ \alpha_i & \text{if } \alpha_i\le\beta_{i-1}\end{cases},i\ge 2;\quad \tilde\beta_i=\begin{cases}[\frac{\alpha_{i+1}+\beta_i+1}{2}] & \text{if } \beta_i<\alpha_{i+1}\\ \beta_i & \text{if } \beta_i\ge\alpha_{i+1}\end{cases},i\ge 1.$$

For $(\alpha,\beta)\in\Lambda^2_{C_n}$, define

$$\tilde\alpha_1=\begin{cases}[\frac{\alpha_1+\beta_1}{2}] & \text{if } \beta_1>\alpha_1+1\\ \alpha_1 & \text{if } \beta_1\le\alpha_1+1\end{cases},\quad \tilde\beta_1=\begin{cases}[\frac{\alpha_1+\beta_1+1}{2}] & \text{if } \beta_1>\alpha_1+1\\ \beta_1 & \text{if } \alpha_2-1\le\beta_1\le\alpha_1+1\\ [\frac{\alpha_2+\beta_1}{2}] & \text{if } \beta_1<\alpha_2-1\end{cases},$$

$$\tilde\alpha_i=\begin{cases}[\frac{\alpha_i+\beta_i}{2}] & \text{if } \beta_i>\alpha_i+1\\ [\frac{\alpha_i+\beta_{i-1}+1}{2}] & \text{if } \beta_{i-1}<\alpha_i-1\\ \alpha_i & \text{if } \beta_i\le\alpha_i+1 \text{ and } \beta_{i-1}\ge\alpha_i-1\end{cases},\ i\ge 2,$$

$$\tilde\beta_i=\begin{cases}[\frac{\alpha_i+\beta_i+1}{2}] & \text{if } \beta_i>\alpha_i+1\\ [\frac{\alpha_{i+1}+\beta_i}{2}] & \text{if } \beta_i<\alpha_{i+1}-1\\ \beta_i & \text{if } \alpha_{i+1}-1\le\beta_i\le\alpha_i+1 \text{ and } \beta_{i-1}\ge\alpha_i-1\\ & \quad\text{or } \beta_i\ge\alpha_{i+1}-1 \text{ and } \beta_{i-1}<\alpha_i-1\end{cases},\ i\ge 2.$$

For $(\alpha,\beta)\in\Lambda^2_{D_n}$, define $\tilde\alpha_1=\alpha_1$ and

$$\tilde\alpha_i=\begin{cases}[\frac{\alpha_i+\beta_{i-1}+2}{2}] & \text{if } \alpha_i>\beta_{i-1}+2\\ \alpha_i & \text{if } \alpha_i\le\beta_{i-1}+2\end{cases},\ i\ge 2,$$

$$\tilde\beta_i=\begin{cases}[\frac{\alpha_{i+1}+\beta_i-1}{2}] & \text{if } \beta_i<\alpha_{i+1}-2\\ \beta_i & \text{if } \beta_i\ge\alpha_{i+1}-2\end{cases},\ i\ge 1,$$

(note that $\Phi((\alpha,\beta))=(\tilde\alpha,\tilde\alpha)$ if and only if $(\alpha,\beta)=(\tilde\alpha,\tilde\alpha)$; we define Φ to be the identity map on the set of degenerate elements of $\mathbf{W}^\wedge$).

It is easy to verify that in each case we get a well-defined element $(\tilde\alpha,\tilde\beta)\in\Lambda^1_G$.

4.2. In this subsection we show that for each $\tilde{c}\in\Omega^1_G$,

(a) $\gamma^p(\Sigma^p_{\tilde{c}})=\Phi^{-1}(\gamma^1_G(\tilde{c}))$.

Then 4.1 (a) follows from (a). In view of Proposition 3.1, (a) follows from

(b) $\Phi|_{\Lambda^1}=Id$ *and* $(\alpha,\beta)\le\Phi(\alpha,\beta)$,

(c) *For any* $(\tilde\alpha',\tilde\beta')\in\Lambda^1$ *such that* $(\alpha,\beta)\le(\tilde\alpha',\tilde\beta')$, *we have* $\Phi(\alpha,\beta)\le(\tilde\alpha',\tilde\beta')$.

The first assertion in (b) follows from the definition of Φ. Suppose $\Phi((\alpha,\beta))=(\tilde\alpha,\tilde\beta)$. Let A_i, B_i, $\tilde A_i,\tilde B_i$, $\tilde A'_i,\tilde B'_i$ be defined for (α,β), $(\tilde\alpha,\tilde\beta)$, $(\tilde\alpha',\tilde\beta')$ respectively as in (1). We prove (b) and (c).

(i) Assume $(\alpha,\beta)\in\Lambda^2_{B_n}$. Note that we have $\tilde\beta_i+\tilde\alpha_{i+1}=\beta_i+\alpha_{i+1}$, $B_1=\tilde B_1$ and thus $B_i=\tilde B_i$. Moreover, $A_i\le\tilde A_i$, and $A_i<\tilde A_i$ if and only if $\beta_i<\alpha_{i+1}$. Hence $(\alpha,\beta)\le(\tilde\alpha,\tilde\beta)$.

Assume there exists $(\tilde\alpha',\tilde\beta')\in\Lambda^1_{B_n}$ such that $(\alpha,\beta)\le(\tilde\alpha',\tilde\beta')$ and $(\tilde\alpha,\tilde\beta)\not\le(\tilde\alpha',\tilde\beta')$. Since $\tilde B_j=B_j\le\tilde B'_j$ for all j, there exists an i such that $\tilde A'_i<\tilde A_i$. It follows that $\beta_i<\alpha_{i+1}$ (since $A_i<\tilde A_i$) and thus $\tilde\beta_i\le\tilde\alpha_{i+1}+1$ by the definition of Φ. On the other hand, $\tilde\beta_i>\tilde\beta'_i\ge\tilde\alpha'_{i+1}>\tilde\alpha_{i+1}$ (we use $\tilde B_j=B_j\le\tilde B'_j$, $j=i,i+1$, and the fact that $(\tilde\alpha',\tilde\beta')\in\Lambda^1_{B_n}$), which is a contradiction.

(ii) Assume $(\alpha,\beta)\in\Lambda^2_{C_n}$. We show by induction on i that

(d) *if* $\beta_i>\alpha_i+1$, *then* $\tilde B_i>B_i$, $\tilde A_i=A_i$*; if* $\alpha_{i+1}-1\leq\beta_i\leq\alpha_i+1$, *then* $\tilde A_i=A_i$, $\tilde B_i=B_i$*; if* $\beta_i<\alpha_{i+1}-1$, *then* $\tilde B_i=B_i$, $\tilde A_i>A_i$.

It then follows that $(\alpha,\beta)\leq(\tilde\alpha,\tilde\beta)$. It is easy to verify that (d) holds when $i=1$. We have the following subcases:

(ii-1) $\beta_{i+1}>\alpha_{i+1}+1$. Then $\tilde\alpha_{i+1}>\alpha_{i+1}$ and $\tilde\alpha_{i+1}+\tilde\beta_{i+1}=\alpha_{i+1}+\beta_{i+1}$. Since $\beta_i>\alpha_{i+1}+1$, by induction hypothesis, $\tilde A_i=A_i$. It follows that $\tilde B_{i+1}>\tilde B_{i+1}$ and $\tilde A_{i+1}=A_{i+1}$.

(ii-2) $\alpha_{i+2}-1\leq\beta_{i+1}\leq\alpha_{i+1}+1$. If $\beta_i\geq\alpha_{i+1}-1$, then $\tilde A_i=A_i$ (by induction hypothesis) and $\tilde\alpha_{i+1}=\alpha_{i+1}$; if $\beta_i<\alpha_{i+1}-1$, then $\tilde B_i=B_i$ (by induction hypothesis) and $\tilde\beta_i+\tilde\alpha_{i+1}=\beta_i+\alpha_{i+1}$. It follows that $\tilde B_{i+1}=B_{i+1}$. Since $\tilde\beta_{i+1}=\beta_{i+1}$, we have $\tilde A_{i+1}=A_{i+1}$.

(ii-3) $\beta_{i+1}<\alpha_{i+2}-1$. If $\beta_i\geq\alpha_{i+1}-1$, then $\tilde A_i=A_i$ and $\tilde\alpha_{i+1}=\alpha_{i+1}$; if $\beta_i<\alpha_{i+1}-1$, then $\tilde B_i=B_i$ and $\tilde\beta_i+\tilde\alpha_{i+1}=\beta_i+\alpha_{i+1}$. It follows that $\tilde B_{i+1}=B_{i+1}$. Since $\tilde\beta_{i+1}>\beta_{i+1}$, we have $\tilde A_{i+1}>A_{i+1}$. (d) is proved.

Assume there exists $(\tilde\alpha',\tilde\beta')\in\Lambda^1_{C_n}$ such that $(\alpha,\beta)\leq(\tilde\alpha',\tilde\beta')$ and $(\tilde\alpha,\tilde\beta)\nleq(\tilde\alpha',\tilde\beta')$. Suppose that there exists an i such that $\tilde A'_i<\tilde A_i$. Then it follows from (d) that $\beta_i<\alpha_{i+1}-1$ (since $A_i<\tilde A_i$) and thus $\tilde\beta_i\leq\tilde\alpha_{i+1}$; $\tilde B_i=B_i$, $\tilde B_{i+1}=B_{i+1}$ and thus $\tilde\beta_i\geq\tilde\beta'_i+1\geq\tilde\alpha'_{i+1}\geq\tilde\alpha_{i+1}+1$, which is a contradiction. Then there exists an i such that $\tilde B'_i<\tilde B_i$. It follows from (d) that $\beta_i>\alpha_i+1$ and thus $\tilde\beta_i\geq\tilde\alpha_i$; $\tilde A_i=A_i\leq\tilde A'_i$, $\tilde A_{i-1}=A_{i-1}\leq\tilde A'_{i-1}$, and thus $\tilde\beta_i\leq\tilde\beta'_i-1\leq\tilde\alpha'_i\leq\tilde\alpha_i-1$, which is again a contradiction.

(iii) Assume $(\alpha,\beta)\in\Lambda^2_{D_n}$. We have $\tilde\beta_i+\tilde\alpha_{i+1}=\beta_i+\alpha_{i+1}$, $B_1=\tilde B_1$ and thus $B_i=\tilde B_i$. Moreover, $A_i\leq\tilde A_i$, and $A_i<\tilde A_i$ if and only if $\beta_i<\alpha_{i+1}-2$. Hence $(\alpha,\beta)\leq(\tilde\alpha,\tilde\beta)$.

Assume there exists $(\tilde\alpha',\tilde\beta')\in\Lambda^1_{D_n}$ such that $(\alpha,\beta)\leq(\tilde\alpha',\tilde\beta')$ and $(\tilde\alpha,\tilde\beta)\nleq(\tilde\alpha',\tilde\beta')$. Then $\tilde B_j=B_j\leq\tilde B'_j$ for all j and there exists an i such that $A_i\leq\tilde A'_i<\tilde A_i$. It follows that $\tilde\beta_i<\tilde\alpha_{i+1}$, and $\tilde\beta_i>\tilde\beta'_i\geq\tilde\alpha'_{i+1}-2>\tilde\alpha_{i+1}-2$, which is a contradiction. This completes the proof of (b) and (c).

5. Explicit description of pieces

5.1. We define a map

$$\Psi^p_G:\Omega^p_G\to\Omega^1_G\ (\text{resp. }\Psi^2_{\mathfrak g}:\Omega^2_{\mathfrak g}\to\Omega^1_G)$$

as follows. Let Ψ^p_G be the natural identification map between Ω^p_G and Ω^1_G if $p\neq 2$.

If $G=Sp(2n)$, let

$$\Psi^2_G((\lambda,\varepsilon))=\lambda\ (\text{resp. }\Psi^2_{\mathfrak g}((\lambda,\chi))=\lambda). \tag{9}$$

If $G=SO(N)$, define $\Psi^2_G((\lambda,\varepsilon))=\tilde\lambda=(\tilde\lambda_1\geq\tilde\lambda_2\geq\cdots)$ as follows, where $\lambda=(\lambda_1\geq\lambda_2\geq\cdots)$. If λ_{2i-1} is even, $\varepsilon(\lambda_{2i-1})=1$, and $\lambda_{2i-1}<\lambda_{2i-2}$ (when $i\geq 2$), then $\tilde\lambda_{2i-1}=\lambda_{2i-1}+1$; if λ_{2i} is even, $\varepsilon(\lambda_{2i})=1$ and $\lambda_{2i}>\lambda_{2i+1}$, then $\tilde\lambda_{2i}=\lambda_{2i}-1$. Otherwise $\tilde\lambda_j=\lambda_j$. Note that $\Psi^2_{SO(2n)}((\lambda,\varepsilon))=\tilde\lambda$ with $\tilde\lambda_i$ all even, if and only if $\lambda=\tilde\lambda$ and $\varepsilon(\lambda_i)=0$ for all i; for the two degenerate classes $\mathrm{c}_1,\mathrm{c}_2$ corresponding to (λ,ε), we define $\Psi^2_G(\mathrm{c}_i)=\tilde{\mathrm{c}}_i$ by $\gamma^2_G(\mathrm{c}_i)=\gamma^1_G(\tilde{\mathrm{c}}_i)$, $i=1,2$.

If $G = SO(N)$ and $(\lambda, \chi) \in \Omega^2_{\mathfrak{g}}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, let $k \geq 0$ be the unique integer such that $\lambda_{2k+2} = \lambda_{2k+1} - 1$ (when N is odd), and $k = \infty$ (when N is even). Define $\Psi^2_{\mathfrak{g}}((\lambda, \chi)) = \tilde{\lambda}$ as follows.

$$\tilde{\lambda}_1 = \begin{cases} 2\chi(\lambda_1) - 1 & \text{if } \chi(\lambda_1) \geq \frac{\lambda_1}{2} + 1 \\ \lambda_1 & \text{if } \chi(\lambda_1) \leq \frac{\lambda_1 + 1}{2} \end{cases},$$

$$\tilde{\lambda}_{2i} = \begin{cases} \lambda_{2i} - \chi(\lambda_{2i}) + \chi(\lambda_{2i+1}) & \text{if } \chi(\lambda_{2i}) \geq \lambda_{2i} - \chi(\lambda_{2i+1}) + 1 \\ & \quad \text{and } i \leq k \\ 2(\lambda_{2i} - \chi(\lambda_{2i})) + 1 & \text{if } \frac{\lambda_{2i}}{2} + 1 \leq \chi(\lambda_{2i}) \leq \lambda_{2i} - \chi(\lambda_{2i+1}) \\ & \quad \text{and } i \leq k \\ \lambda_{2i} & \text{if } \chi(\lambda_{2i}) \leq \frac{\lambda_{2i}+1}{2} \text{ and } i \leq k \\ \lambda_{2i+1} & \text{if } i \geq k+1 \end{cases},$$

$$\tilde{\lambda}_{2i+1} = \begin{cases} \lambda_{2i} - \chi(\lambda_{2i}) + \chi(\lambda_{2i+1}) & \text{if } \chi(\lambda_{2i+1}) \geq \lambda_{2i} - \chi(\lambda_{2i}) + 1 \\ & \quad \text{and } i \leq k \\ 2\chi(\lambda_{2i+1}) - 1 & \text{if } i \leq k \text{ and} \\ & \frac{\lambda_{2i+1}}{2} + 1 \leq \chi(\lambda_{2i+1}) \leq \lambda_{2i} - \chi(\lambda_{2i}) \\ \lambda_{2i+1} & \text{if } \chi(\lambda_{2i+1}) \leq \frac{\lambda_{2i+1}+1}{2} \text{ or } i > k \end{cases}, \quad i \geq 1.$$

Note that $\Psi^2_{\mathfrak{so}(2n)}((\lambda, \chi)) = \tilde{\lambda}$ with $\tilde{\lambda}_i$ all even, if and only if $\lambda = \tilde{\lambda}$ and $\chi(\lambda_i) = \lambda_i/2$ for all i; for the two degenerate classes $\mathrm{c}_1, \mathrm{c}_2$ corresponding to (λ, χ), we define $\Psi^2_{\mathfrak{g}}(\mathrm{c}_i) = \tilde{\mathrm{c}}_i$ by $\gamma^2_{\mathfrak{g}}(\mathrm{c}_i) = \gamma^1_G(\tilde{\mathrm{c}}_i)$, $i = 1, 2$.

Using the definition of Ψ^p_G (resp. $\Psi^2_{\mathfrak{g}}$) and the description of the maps γ^p in 2.4, 2.5 and 2.6, one verifies that

(a) $\gamma^1_G \circ \Psi^p_G = \Phi \circ \gamma^p_G$ *(resp.* $\gamma^1_G \circ \Psi^2_{\mathfrak{g}} = \Phi \circ \gamma^2_{\mathfrak{g}}$).

Proposition 5.1. *Two classes* $\mathrm{c}, \mathrm{c}' \in \Omega^p_G$ *(resp.* $\Omega^2_{\mathfrak{g}}$*) lie in the same unipotent (resp. nilpotent) piece as defined in* [7] *if and only if* $\Psi^p_G(\mathrm{c}) = \Psi^p_G(\mathrm{c}')$ *(resp.* $\Psi^2_{\mathfrak{g}}(\mathrm{c}) = \Psi^2_{\mathfrak{g}}(\mathrm{c}')$*).*

Proposition is clear when $p \neq 2$. If $G = Sp(2n)$, the proposition follows from [7] (see Lemma 2.7 and (9)). The case where $p = 2$ and $G = SO(N)$ is proved in subsection 5.2. Note that the proposition computes the pieces in classical groups explicitly. Another computation of the unipotent pieces is given in [9]. Now in view of (a) and 4.2 (a), it follows from Proposition 5.1 that

(b) *each set* $\Sigma^p_{\tilde{\mathrm{c}}}$, $\tilde{\mathrm{c}} \in \Omega^1_G$*, defined in 4.1 is an unipotent (resp. a nilpotent) piece defined in* [7].

5.2. Let $G = SO(V)$ be as in 2.8. We prove Proposition 5.1 by induction on $\dim V$. Assume $\Psi^2_G(\mathrm{c}) = \tilde{\mathrm{c}}$ (resp. $\Psi^2_{\mathfrak{g}}(\mathrm{c}) = \tilde{\mathrm{c}}$). We show that

(a) $\Upsilon_{\mathrm{c}} = \Upsilon_{\tilde{\mathrm{c}}}$.

Suppose that $\Upsilon_{\mathrm{c}} = \{f_a\}, \Upsilon_{\tilde{\mathrm{c}}} = \{\tilde{f}_a\}$. If $G = SO(2n)$, then it follows from (a) that c is degenerate if and only if $f_0 = 0$, and then from [7] that each degenerate class itself is a piece. Now Proposition 5.1 follows from Lemma 2.8. If $\mathrm{c} = \{0\}$, (a) is obvious. We assume $\mathrm{c} \neq \{0\}$. Let $u \in \mathrm{c}$ (resp. $x \in \mathrm{c}$) and let V', u' (resp. x') be as in 2.8. Let c' be the class of u' (resp. x') in $G' = SO(V')$ (resp. $\mathfrak{g}' = \mathfrak{so}(V')$) and let $\tilde{\mathrm{c}}' = \Psi^2_{G'}(\mathrm{c}')$ (resp. $\Psi^2_{\mathfrak{g}'}(\mathrm{c}')$). Suppose that $\Upsilon_{\mathrm{c}'} = \{f'_a\}, \Upsilon_{\tilde{\mathrm{c}}'} = \{\tilde{f}'_a\}$. Let m, e, f be defined for $T = u - 1$ (resp. x) as in 2.8. Assume $\tilde{\mathrm{c}} = \tilde{\lambda}$, where $\tilde{\lambda} = (\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots)$. We show that

(b) $m=\tilde{\lambda}_1-1$, $\tilde{f}_a=\tilde{f}'_a$ *for all* $a\in[-m+1,m-1]$, *and* $f_m=\tilde{f}_m$.

Since $\dim V'<\dim V$, by induction hypothesis, $\Upsilon_{\mathrm{c}'}=\Upsilon_{\tilde{\mathrm{c}}'}$. It then follows from (b) and 2.8 (b) that $f_a=\tilde{f}_a$ for all a, since $f_a=f_{-a}=\tilde{f}_a=\tilde{f}_{-a}=0$ for all $a\geq m+1$. Hence (a) holds.

We prove (b) for Ψ^2_G. Assume $\mathrm{c}=(\lambda,\varepsilon)$, $\mathrm{c}'=(\lambda',\varepsilon')$, and $\tilde{\mathrm{c}}'=\tilde{\lambda}'$. Using definition of Ψ^2_G and 2.8, we can compute c', $\tilde{\mathrm{c}}$, $\tilde{\mathrm{c}}'$, $f_m=\dim V_{\geq m}=\dim(V_{\geq -m+1}\cap Q^{-1}(0))$ and $\tilde{f}_m=m_{\tilde{\lambda}}(\tilde{\lambda}_1)$ in various cases as follows.

(I) $\varepsilon(\lambda_1)=1$ and $m_\lambda(\lambda_1)=2m_1$. We have $e=\lambda_1=2f-2$ and thus $m=\lambda_1$, $f_m=1$; $\lambda'_i=\lambda_1$, $i\in[1,2m_1-2]$, $\varepsilon'(\lambda_1)\leq 0$, $\lambda'_{2m_1-1}=\lambda'_{2m_1}=\lambda_1-1$, and $\lambda'_j=\lambda_j$, $\varepsilon'(\lambda'_j)=\varepsilon(\lambda_j)$ for all $j\geq 2m_1+1$. Moreover $\tilde{\lambda}_1=\lambda_1+1$, $\tilde{\lambda}_i=\lambda_1$, $i\in[2,2m_1-1]$ and $\tilde{\lambda}_{2m_1}=\lambda_1-1$; $\tilde{\lambda}'_i=\lambda_1$, $i\in[1,2m_1-2]$, $\tilde{\lambda}'_{2m_1-1}=\lambda_1-1$ and $\tilde{\lambda}'_j=\tilde{\lambda}_j$, $j\geq 2m_1$. Thus $\tilde{f}_m=1$.

(II) $\varepsilon(\lambda_1)=0$ and $m_\lambda(\lambda_1)=2m_1$. We have $e=\lambda_1=2f$ and thus $m=\lambda_1-1$, $f_m=2m_1$. Let $m_\lambda(\lambda_1-1)=2m_2\geq 0$. We have $\tilde{\lambda}_i=\lambda_1$, $i\in[1,2m_1]$, $\tilde{\lambda}_{2m_1+i}=\lambda_1-1$, $i\in[1,2m_2]$. We have the following cases:

(i) $\varepsilon(\lambda_1-2)<1$. Then $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$, $\varepsilon'(\lambda_1-2)=0$, and $\lambda'_j=\lambda_j$, $\varepsilon'(\lambda'_j)=\varepsilon(\lambda_j)$ for all $j\geq 2m_1+2m_2+1$. Moreover, $\tilde{\lambda}'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde{\lambda}'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$, and $\tilde{\lambda}'_j=\tilde{\lambda}_j$, $j\geq 2m_1+2m_2+1$. Thus $\tilde{f}_m=2m_1$.

(ii) $\varepsilon(\lambda_1-2)=1$. Then $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+1]$, $\varepsilon'(\lambda_1-2)=1$, and $\lambda'_j=\lambda_j$, $\varepsilon'(\lambda'_j)=\varepsilon(\lambda_j)$ for all $j\geq 2m_1+2m_2+2$. Moreover, $\tilde{\lambda}_{2m_1+2m_2+1}=\lambda_1-1$, thus $\tilde{f}_m=2m_1$; $\tilde{\lambda}'_i=\lambda_1-1$, $i\in[1,2m_2+1]$, $\tilde{\lambda}'_{2m_2+1+i}=\lambda_1-2$, $i\in[1,2m_1]$ and $\tilde{\lambda}'_j=\tilde{\lambda}_j$, $j\geq 2m_1+2m_2+2$.

(III) $m_\lambda(\lambda_1)=2m_1+1$. We have $e=\lambda_1=2f-2$ and thus $m=\lambda_1$, $f_m=1$. Moreover, $\tilde{\lambda}_1=\lambda_1+1$, $\tilde{\lambda}_i=\lambda_1$, $i\in[2,2m_1+1]$. Let $m_\lambda(\lambda_1-1)=2m_2\geq 0$. Then $\lambda'_i=\lambda_1$, $\varepsilon'(\lambda_1)\leq 0$, $i\in[1,2m_1]$, $\lambda'_{2m_1+i}=\lambda_1-1$, $i\in[1,2m_2]$, $\lambda'_{2m_1+2m_2+1}=\lambda_1-2$, $\varepsilon'(\lambda_1-2)=1$, and $\lambda'_j=\lambda_j$, $\varepsilon(\lambda'_j)=\varepsilon(\lambda_j)$ for all $j\geq 2m_1+2m_2+2$. Moreover $\tilde{\lambda}_{2m_1+1+i}=\lambda_1-1$, $i\in[1,2m_2]$; $\tilde{\lambda}'_i=\lambda_1$, $i\in[1,2m_1]$, $\tilde{\lambda}'_{2m_1+1}=\lambda_1-1$, and $\tilde{\lambda}'_j=\tilde{\lambda}_j$, $j\geq 2m_1+2$. Thus $\tilde{f}_m=1$.

(IV) $\varepsilon(\lambda_1)=\omega$. Then $m_\lambda(\lambda_1)=2m_1$. We have $e=\lambda_1=2f-1$ and thus $m=\lambda_1-1$. Moreover, $\tilde{\lambda}_i=\lambda_1$, $i\in[1,2m_1]$. We have the following cases:

(i) $m_\lambda(\lambda_1-1)=2m_2$ and $\varepsilon(\lambda_1-1)=1$ (note then $m_2>0$). Then $f_m=2m_1+1$; $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2-2]$, $\varepsilon'(\lambda_1-1)\leq 0$, $\lambda'_{2m_2-2+i}=\lambda_1-2$, $i\in[1,2m_1+2]$, and $\lambda'_j=\lambda_j$, $\varepsilon'(\lambda'_j)=\varepsilon(\lambda_j)$ for all $j\geq 2m_1+2m_2+1$. Moreover $\tilde{\lambda}_{2m_1+1}=\lambda_1$, $\tilde{\lambda}_{2m_1+1+i}=\lambda_1-1$, $i\in[1,2m_2-2]$, and $\tilde{\lambda}_{2m_1+2m_2}=\lambda_1-2$, thus $\tilde{f}_m=2m_1+1$; $\tilde{\lambda}'_i=\lambda_1-1$, $i\in[1,2m_2-2]$, $\tilde{\lambda}'_{2m_2-2+i}=\lambda_1-2$, $i\in[1,2m_1+1]$ and $\tilde{\lambda}'_j=\tilde{\lambda}_j$, $j\geq 2m_1+2m_2$.

(ii) $m_\lambda(\lambda_1-1)=2m_2$ and $\varepsilon(\lambda_1-1)\leq 0$. Then $f_m=2m_1$; $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\varepsilon'(\lambda_1-1)\leq 0$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$, and $\lambda'_j=\lambda_j$, $\varepsilon'(\lambda'_j)=\varepsilon(\lambda_j)$ for all $j\geq 2m_1+2m_2+1$. Moreover $\tilde{\lambda}_{2m_1+i}=\lambda_1-1$, $i\in[1,2m_2]$; $\tilde{\lambda}'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde{\lambda}'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$ and $\tilde{\lambda}'_j=\tilde{\lambda}_j$, $j\geq 2m_1+2m_2+1$. Thus $\tilde{f}_m=2m_1$.

(iii) $m_\lambda(\lambda_1-1)=2m_2+1$. Then $f_m=2m_1+1$; $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\varepsilon'(\lambda_1-1)\leq 0$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$, $\lambda_{2m_1+2m_2+1}=\lambda_1-3$, $\varepsilon'(\lambda_1-3)=1$, and $\lambda'_j=\lambda_j$, $\varepsilon'(\lambda'_j)=\varepsilon(\lambda_j)$

for all $j \geq 2m_1+2m_2+2$. Moreover $\tilde\lambda_{2m_1+1}=\lambda_1$, $\tilde\lambda_{2m_1+1+i}=\lambda_1-1$, $i\in[1,2m_2]$; $\tilde\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+1]$ and $\tilde\lambda'_j=\tilde\lambda_j$, $j\geq 2m_1+2m_2+2$. Thus $\tilde f_m=2m_1+1$.

In each case it is now easy to see that (b) holds.

We prove (b) for $\Psi^2_{\mathfrak g}$. Assume $\mathrm{c}=(\lambda,\chi)$, $\mathrm{c}'=(\lambda',\chi')$, and $\tilde{\mathrm{c}}'=\tilde\lambda'$. For $(\lambda,\chi)\in\Omega^2_{\mathfrak g}$, we extend χ to a function $\chi:\mathbb N\to\mathbb N$ as follows:

$$\chi(i)=\max_j([\lambda_j;\chi(\lambda_j)](i)),$$

where $[p;l](i)=\max(0,\min(i-p+l,l))$.

Using definition of $\Psi^2_{\mathfrak g}$ and 2.8, we compute $\mathrm{c}',\tilde{\mathrm{c}},\tilde{\mathrm{c}}'$, $f_m=\dim V_{\geq m}=\dim(V_{\geq -m+1}\cap Q^{-1}(0))$ and $\tilde f_m=m_{\tilde\lambda}(\tilde\lambda_1)$ in various cases as follows.

(I) $\chi(\lambda_1)=\lambda_1/2$ and $\lambda_1=2$. Write $m_\lambda(2)=2m_1$ and $m_\lambda(1)=m_2$. We have $e=2$, $f=1$ and thus $m=1$, $f_m=2m_1$. Then $\lambda'_i=1, i\in[1,m_2]$, $\lambda'_i=0$ for all $i\geq m_2+1$. Moreover, $\tilde\lambda_i=2$, $i\in[1,2m_1]$ and $\tilde\lambda_{2m_1+i}=1$, $i\in[1,m_2]$; $\tilde\lambda'_i=1$, $i\in[1,m_2]$, $\tilde\lambda'_i=0$, $i\geq m_2+1$. Thus $\tilde f_m=2m_1$.

(II) $\chi(\lambda_1)=\lambda_1/2$ and $\lambda_1\geq 4$. Then $m_\lambda(\lambda_1)=2m_1$ and $m_\lambda(\lambda_1-1)=2m_2$ ($m_2\geq 0$). We have $e=\lambda_1$, $f=\lambda_1/2$ and thus $m=\lambda_1-1$, $f_m=2m_1$. Moreover, $\tilde\lambda_i=\lambda_1$, $i\in[1,2m_1]$, $\tilde\lambda_{2m_1+i}=\lambda_1-1$, $i\in[1,2m_2]$. We have the following cases:

(i) $\chi(\lambda_1-2)=\lambda_1/2$. Then $\lambda'_i=\lambda_1-1$, $\chi'(\lambda_1-1)=\lambda_1/2$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $\chi'(\lambda_1-2)=\lambda_1/2$, $i\in[1,2m_1+1]$. Moreover, $\tilde\lambda_{2m_1+2m_2+1}=\lambda_1-1$, thus $\tilde f_m=2m_1$; $\tilde\lambda'_i=\lambda_1-1$, $i\in[1,2m_2+1]$, $\tilde\lambda'_{2m_2+1+i}=\lambda_1-2$, $i\in[1,2m_1]$ and $\tilde\lambda'_j=\tilde\lambda_j$ for all $j\geq 2m_1+2m_2+2$.

(ii) $\chi(\lambda_1-2)<\lambda_1/2$. Then $\lambda'_i=\lambda_1-1$, $\chi'(\lambda_1-1)=\lambda_1/2$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $\chi'(\lambda_1-2)=\lambda_1/2-1$, $i\in[1,2m_1]$. Moreover, $\tilde\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$ and $\tilde\lambda'_j=\tilde\lambda_j$ for all $j\geq 2m_1+2m_2+1$. Thus $\tilde f_m=2m_1$.

(III) $\chi(\lambda_1)=\frac{\lambda_1+1}{2}$. Then $m_\lambda(\lambda_1)=2m_1$. We have $e=\lambda_1, f=\frac{\lambda_1+1}{2}$ and thus $m=\lambda_1-1$; $\tilde\lambda_i=\lambda_1$, $i\in[1,2m_1]$. There exists a unique $j\geq 0$ such that $\chi(\lambda_1-j)=\frac{\lambda_1+1}{2}$ and $\chi(\lambda_1-j-1)<\frac{\lambda_1+1}{2}$. Assume $m_\lambda(\lambda_1-i)=2m_{i+1}$, $i\in[1,j-1]$. We have the following subcases:

(i) $j=0$. Then $\chi(\lambda_1-1)=\frac{\lambda_1-1}{2}$, $m_\lambda(\lambda_1-1)=2m_2$ (since $\frac{\lambda_1-1}{2}<\lambda_1-1$), and $f_m=2m_1$. We have $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$, $\chi'(\lambda_1-k)=\frac{\lambda_1-1}{2}$, $k\in[1,2]$, and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq 2m_1+2m_2+1$. Moreover, $\tilde\lambda_{2m_1+i}=\lambda_1-1$, $i\in[1,2m_2]$; $\tilde\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1]$ and $\tilde\lambda'_i=\tilde\lambda_i$ for all $i\geq 2m_1+2m_2+1$; thus $\tilde f_m=2m_1$.

(ii) $j=1$ and $m_\lambda(\lambda_1-1)=2m_2$. Then $m_2>0$ and $f_m=2m_1+1$. We have $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2-2]$, $\lambda'_{2m_2-2+i}=\lambda_1-2$, $i\in[1,2m_1+2]$, $\chi'(\lambda_1-k)=\frac{\lambda_1-1}{2}$, $k\in[1,2]$, and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq 2m_1+2m_2+1$. Moreover, $\tilde\lambda_{2m_1+1}=\lambda_1$, $\tilde\lambda_{2m_1+1+i}=\lambda_1-1$, $i\in[1,2m_2-2]$; $\tilde\lambda'_i=\lambda_1-1$, $i\in[1,2m_2-2]$, $\tilde\lambda'_{2m_2-2+i}=\lambda_1-2$, $i\in[1,2m_1+1]$ and $\tilde\lambda'_i=\tilde\lambda_i$ for all $i\geq 2m_1+2m_2$. Thus $\tilde f_m=2m_1+1$.

(iii) $j=2$ and $m(\lambda_1-2)=2m_3$. Then $m_3>0$ and $f_m=2m_1+1$. We have $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+2m_3-2]$, $\lambda'_{2m_1+2m_2+2m_3-1}=\lambda'_{2m_1+2m_2+2m_3}=\lambda_1-3$, $\chi'(\lambda_1-k)=\frac{\lambda_1-1}{2}$, $k\in[1,3]$, and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq 2m_1+2m_2+2m_3+1$.

Moreover, $\tilde{\lambda}_{2m_1+1}=\lambda_1$, $\tilde{\lambda}_{2m_1+1+i}=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde{\lambda}_{2m_1+2m_2+1+i}=\lambda_1-2$, $i\in[1,2m_3-2]$; $\tilde{\lambda}'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde{\lambda}'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+1]$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1+2m_2+2$; thus $\tilde{f}_m=2m_1+1$.

(iv) $j\geq 3$ and $m_\lambda(\lambda_1-j)=2m_{j+1}$. We have $f_m=2m_1+1$; $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+2m_3]$, $\lambda'_{\sum_{a\in[1,k]}2m_a+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[3,j-1]$, $\lambda'_{\sum_{a\in[1,j]}2m_a+i}=\lambda_1-j$, $i\in[1,2m_{j+1}-2]$, $\lambda'_{\sum_{a\in[1,j+1]}2m_a-2+i}=\lambda_1-j-1$, $i\in[1,2]$, $\chi'(\lambda_1-k)=\frac{\lambda_1-1}{2}$, $k\in[1,j+1]$, and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq\sum_{a\in[1,j+1]}2m_a+1$. Moreover, $\tilde{\lambda}_{2m_1+1}=\lambda_1$, $\tilde{\lambda}_{\sum_{a\in[1,k]}2m_a+1+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[1,j-1]$, $\tilde{\lambda}_{\sum_{a\in[1,j]}2m_a+1+i}=\lambda_1-j$, $i\in[1,2m_{j+1}-2]$; $\tilde{\lambda}'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde{\lambda}'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+1]$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1+2m_2+2$. Thus $\tilde{f}_m=2m_1+1$.

(v) $j=1$ and $m_\lambda(\lambda_1-1)=2m_2+1$. It follows that $\lambda_1=3$ and $m_\lambda(1)=2m_3+1$. We have $f_m=2m_1+1$; $\lambda'_i=2$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=1$, $i\in[1,2m_1+2m_3+1]$, $\chi'(2)=1$, $\lambda'_i=0$, for all $i\geq\sum_{a\in[1,3]}2m_a+2$. Moreover, $\tilde{\lambda}_i=3$, $i\in[1,2m_1+1]$, $\tilde{\lambda}_{2m_1+1+i}=2$, $i\in[1,2m_2]$, $\tilde{\lambda}_{2m_1+2m_2+1+i}=1$, $i\in[1,2m_3]$; $\tilde{\lambda}'_i=2$, $i\in[1,2m_2]$, $\tilde{\lambda}'_{2m_2+i}=1$, $i\in[1,2m_1+1]$ and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1+2m_2+2$. Thus $\tilde{f}_m=2m_1+1$.

(vi) $j\geq 2$ and $m_\lambda(\lambda_1-j)=2m_{j+1}+1$. It follows that $\lambda_1-j=\frac{\lambda_1+1}{2}$ and $m_\lambda(\lambda_1-j-1)=2m_{j+2}+1$. We have $f_m=2m_1+1$; $\lambda'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\lambda'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+2m_3]$, $\lambda'_{\sum_{a\in[1,k]}2m_a+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[3,j]$, $\lambda'_{\sum_{a\in[1,j+1]}2m_a+i}=\lambda_1-j-1$, $i\in[1,2m_{j+2}+1]$, $\lambda'_{\sum_{a\in[1,j+2]}2m_a+2}=\lambda_1-j-2$, $\chi'(\lambda_1-k)=\frac{\lambda_1-1}{2}$, $k\in[1,j+1]$, $\chi'(\lambda_1-j-2)=\frac{\lambda_1-3}{2}$ and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq\sum_{a\in[1,j+2]}2m_a+3$. Moreover, $\tilde{\lambda}_{2m_1+1}=\lambda_1$, $\tilde{\lambda}_{\sum_{a\in[1,k]}2m_a+1+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[1,j+1]$; $\tilde{\lambda}'_i=\lambda_1-1$, $i\in[1,2m_2]$, $\tilde{\lambda}'_{2m_2+i}=\lambda_1-2$, $i\in[1,2m_1+1]$ and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1+2m_2+2$; thus $\tilde{f}_m=2m_1+1$.

(IV) $\chi(\lambda_1)>\frac{\lambda_1+1}{2}$ and $m_\lambda(\lambda_1)=2m_1$. There exists a unique $j\geq 0$ such that $\chi(\lambda_1-j)=\chi(\lambda_1)$ and $\chi(\lambda_1-j-1)<\chi(\lambda_1)$. Assume $m_\lambda(\lambda_1-i)=2m_{i+1}$, $i\in[1,j-1]$. We have $e=\lambda_1$, $f=\chi(\lambda_1)$ and thus $m=2\chi(\lambda_1)-2$, $f_m=1$.

(i) $j=0$. We have $\lambda'_i=\lambda_1$, $i\in[1,2m_1-2]$, $\lambda'_{2m_1-1}=\lambda'_{2m_1}=\lambda_1-1$, $\chi'(\lambda_1)=\chi'(\lambda_1-1)=\chi(\lambda_1)-1$, and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq 2m_1+1$. Moreover, $\tilde{\lambda}_1=2\chi(\lambda_1)-1$, $\tilde{\lambda}_i=\lambda_1$, $i\in[2,2m_1-1]$. If $\chi(\lambda_1)=\frac{\lambda_1+2}{2}$, then $\tilde{\lambda}'_i=\lambda_1$, $i\in[1,2m_1-2]$, $\tilde{\lambda}'_{2m_1-1}=\lambda_1-1$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1$; if $\chi(\lambda_1)\geq\frac{\lambda_1+3}{2}$, then $\tilde{\lambda}'_1=2\chi(\lambda_1)-3$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2$. Thus $\tilde{f}_m=1$.

(ii) $j\geq 1$ and $m_\lambda(\lambda_1-j)=2m_{j+1}$. We have $\lambda'_{\sum_{a\in[1,k]}2m_a+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[0,j-1]$, $\lambda'_{\sum_{a\in[1,j]}2m_a+i}=\lambda_1-j$, $i\in[1,2m_{j+1}-2]$, $\lambda'_{\sum_{a\in[1,j+1]}2m_a-2+i}=\lambda_1-j-1$, $i\in[1,2]$, $\chi'(\lambda_1-k)=\chi(\lambda_1)-1$, $k\in[0,j+1]$, and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq\sum_{a\in[1,j+1]}2m_a+1$. Moreover, $\tilde{\lambda}_1=2\chi(\lambda_1)-1$, $\tilde{\lambda}_{\sum_{a\in[1,k]}2m_a+1+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[0,j-1]$, $\tilde{\lambda}_{\sum_{a\in[1,j]}2m_a+1+i}=\lambda_1-j$, $i\in[1,2m_{j+1}-2]$. If $\chi(\lambda_1)=\frac{\lambda_1+2}{2}$, then $\tilde{\lambda}'_i=\lambda_1$, $i\in[1,2m_1]$, $\tilde{\lambda}'_{2m_1+1}=\lambda_1-1$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1+2$; if $\chi(\lambda_1)\geq\frac{\lambda_1+3}{2}$, then $\tilde{\lambda}'_1=2\chi(\lambda_1)-3$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2$. Thus $\tilde{f}_m=1$.

(iii) $j\geq 1$ and $m_\lambda(\lambda_1-j)=2m_{j+1}+1$. It follows that $\lambda_1-j=\chi(\lambda_1)$ and $m_\lambda(\lambda_1-j-1)=2m_{j+2}+1$. We have $\lambda'_{\sum_{a\in[1,k]}2m_a+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[0,j]$, $\lambda'_{\sum_{a\in[1,j+1]}2m_a+i}=\lambda_1-j-1$, $i\in[1,2m_{j+2}+1]$, $\lambda'_{\sum_{a\in[1,j+2]}2m_a+2}=\lambda_1-j-2$, $\chi'(\lambda_1-k)=\chi(\lambda_1)-1$, $k\in[0,j+1]$, $\chi'(\lambda_1-j-2)=$

$\chi(\lambda_1)-2$, and $\lambda'_i=\lambda_i$, $\chi'(\lambda'_i)=\chi(\lambda_i)$, for all $i\geq\sum_{a\in[1,j+2]}2m_a+3$. Moreover, $\tilde{\lambda}_1=2\chi(\lambda_1)-1$, $\tilde{\lambda}_{\sum_{a\in[1,k]}2m_a+1+i}=\lambda_1-k$, $i\in[1,2m_{k+1}]$, $k\in[0,j+1]$. If $\chi(\lambda_1)=\frac{\lambda_1+2}{2}$, $\tilde{\lambda}'_i=\lambda_1$, $i\in[1,2m_1]$, $\tilde{\lambda}'_{2m_1+1}=\lambda_1-1$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1+2$; if $\chi(\lambda_1)\geq\frac{\lambda_1+3}{2}$, $\tilde{\lambda}'_1=2\chi(\lambda_1)-3$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2$. Thus $\tilde{f}_m=1$.

(V) $m_\lambda(\lambda_1)=2m_1+1$, then $\chi(\lambda_1)=\lambda_1>\frac{\lambda_1+1}{2}$ and $m_\lambda(\lambda_1-1)=2m_2+1$. We have $e=f=\lambda_1$ and thus $m=2\lambda_1-2$, $f_m=1$. Moreover, $\tilde{\lambda}_1=2\lambda_1-1$, $\tilde{\lambda}_i=\lambda_1$, $i\in[2,2m_1+1]$. We have $\lambda'_i=\lambda_1$, $i\in[1,2m_1]$, $\lambda'_{2m_1+i}=\lambda_1-1$, $i\in[1,2m_2+1]$, $\lambda'_{2m_1+2m_2+2}=\lambda_1-2$, $\chi'(\lambda_1)=\chi'(\lambda_1-1)=\lambda_1-1$, $\chi'(\lambda_1-2)=\lambda_1-2$, $\lambda'_i=\lambda_i=\chi(\lambda_i)$ for all $i\geq 2m_1+2m_2+3$. Moreover, if $\lambda_1=2$, then $\tilde{\lambda}'_i=\lambda_1$, $i\in[1,2m_1]$, $\tilde{\lambda}'_{2m_1+1}=\lambda_1-1$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2m_1+2$; if $\lambda_1\geq 3$, then $\tilde{\lambda}'_1=2\lambda_1-3$, and $\tilde{\lambda}'_i=\tilde{\lambda}_i$ for all $i\geq 2$. Thus $\tilde{f}_m=1$.

In each case it is now easy to see that (b) holds. Proposition 5.1 is proved.

6. SPECIAL PIECES

We say that a unipotent or nilpotent class c is special if $\gamma^p(\mathrm{c})$ is a special character of $\mathbf{W}$ (see [2, 4]). If G is of type B_n or C_n, then $(\alpha,\beta)\in\mathcal{P}_2(n)$ is special if and only if $\alpha_{i+1}\leq\beta_i\leq\alpha_i+1$ for all $i\geq 1$; if G is of type D_n, then $(\alpha,\beta)\in\mathbf{W}^\wedge$ is special if and only if $\alpha_{i+1}-1\leq\beta_i\leq\alpha_i$ for all $i\geq 1$, in particular, each degenerate character is special (see [2]).

Let c be a special unipotent (resp. nilpotent) class in G (resp. $\mathfrak{g}$). We define the corresponding special piece $\mathcal{S}_\mathrm{c}$ to be the subset of $\mathcal{U}_G$ (resp. $\mathcal{N}_\mathfrak{g}$) consisting of all elements in the closure of c which are not in the closure of any special unipotent (resp. nilpotent) class $\mathrm{c}'<\mathrm{c}$ (see [4] when $p=1$). We show that a special piece is a union of unipotent (resp. nilpotent) pieces (for unipotent case, see also [9]). Hence $\mathcal{U}_G$ (resp. $\mathcal{N}_\mathfrak{g}$) is partitioned into special pieces $\mathcal{S}_\mathrm{c}$ indexed by special unipotent (resp. nilpotent) classes c (when $p\neq 2$, this follows from [4]). In the remainder of this subsection assume $p=2$.

Let c be a special class and let $\mathcal{S}_\mathrm{c}$ be the corresponding special piece. Let $\tilde{\mathrm{c}}\in\Omega^1_G$ be such that $\gamma^2(\mathrm{c})=\gamma^1_G(\tilde{\mathrm{c}})$. Assume the corresponding special piece $\mathcal{S}_{\tilde{\mathrm{c}}}$ (in the unipotent variety of the group over $\mathbb{C}$ of the same type as G) is a union of the special class $\tilde{\mathrm{c}}:=\tilde{\mathrm{c}}^0$ and non-special classes $\tilde{\mathrm{c}}^1,\ldots,\tilde{\mathrm{c}}^m$. We show that

$$\text{(a) } \mathcal{S}_\mathrm{c}=\sqcup_{i\in[0,m]}\Sigma^2_{\tilde{\mathrm{c}}^i}.$$

Assume $\gamma^1_G(\tilde{\mathrm{c}})=(\tilde{\alpha},\tilde{\beta})$. Let $\mathrm{c}^*\in\mathcal{S}_\mathrm{c}$ and assume $\gamma^2(\mathrm{c}^*)=(\alpha,\beta)$. Then $(\alpha,\beta)\leq(\tilde{\alpha},\tilde{\beta})$ and for any special $(\tilde{\alpha}',\tilde{\beta}')<(\tilde{\alpha},\tilde{\beta})$, $(\alpha,\beta)\not\leq(\tilde{\alpha}',\tilde{\beta}')$. Assume $\Phi(\alpha,\beta)=(\tilde{\alpha}^*,\tilde{\beta}^*)$. It follows from 4.2 (b) that $(\tilde{\alpha}^*,\tilde{\beta}^*)\not\leq(\tilde{\alpha}',\tilde{\beta}')$ and from 4.2 (c) that $(\tilde{\alpha}^*,\tilde{\beta}^*)\leq(\tilde{\alpha},\tilde{\beta})$. Hence $(\tilde{\alpha}^*,\tilde{\beta}^*)=(\tilde{\alpha}^i,\tilde{\beta}^i)$ for some $i\in[0,m]$ and thus $\mathrm{c}^*\in\Sigma^2_{\tilde{\mathrm{c}}^i}$ (note if c is a degenerate class, then $m=0$ (see [2, 4]) and $\mathrm{c}^*=\mathrm{c}$). This shows that $\mathcal{S}_\mathrm{c}\subset\sqcup_{i\in[0,m]}\Sigma^2_{\tilde{\mathrm{c}}^i}$.

Now if c is a degenerate class, then $m=0$ and r.h.s of (a) is $\{\mathrm{c}\}\subset\mathcal{S}_\mathrm{c}$ (see 4.2 (a)). Assume c is not a degenerate class and assume $\gamma^1_G(\tilde{\mathrm{c}}^j)=(\tilde{\alpha}^j,\tilde{\beta}^j)$, $j\in[0,m]$. Assume $\mathrm{c}^*\in\Sigma^2_{\tilde{\mathrm{c}}^j}$ and $\gamma^2(\mathrm{c}^*)=(\alpha^j,\beta^j)$. Then we have $\Phi(\alpha^j,\beta^j)=(\tilde{\alpha}^j,\tilde{\beta}^j)$ (see 4.2 (a)). Let $\mathrm{c}'<\mathrm{c}$ be another special class and assume that $\gamma^2(\mathrm{c}')=(\tilde{\alpha}',\tilde{\beta}')$. We have $(\alpha^j,\beta^j)\leq(\tilde{\alpha}^j,\tilde{\beta}^j)\leq(\tilde{\alpha},\tilde{\beta})$ (see 4.2 (b)) and $(\alpha^j,\beta^j)\not\leq(\tilde{\alpha}',\tilde{\beta}')$ (see 4.2 (c)). Thus $\mathrm{c}^*\in\mathcal{S}_\mathrm{c}$. Hence $\Sigma^2_{\tilde{\mathrm{c}}^i}\subset\mathcal{S}_\mathrm{c}$, $i\in[0,m]$. The proof of (a) is completed.

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References

[1] W.H. Hesselink, Nilpotency in classical groups over a field of characteristic 2. Math. Z. 166 (1979), 165-181.
[2] G. Lusztig, A class of irreducible representations of a Weyl group. Nederl. Akad. Wetensch. Indag. Math. 41 (1979), no. 3, 323-335.
[3] G. Lusztig, Intersection cohomology complexes on a reductive group. Invent. Math. 75 (1984), no.2, 205-272.
[4] G. Lusztig, Notes on unipotent classes. Asian J. Math. 1 (1997), no. 1, 194-207.
[5] G. Lusztig, Unipotent elements in small characteristic. Transform.Groups 10 (2005), 449-487.
[6] G. Lusztig, Unipotent elements in small characteristic, II. Transform.Groups 13 (2008), 773-797.
[7] G. Lusztig, Unipotent elements in small characteristic, III. doi:10.1016/j.algebra.2009.12.008.
[8] G. Lusztig, Remarks on Springer's representations. Represent.Th. 13 (2009), 391-400 (electronic).
[9] G. Lusztig, From conjugacy classes in the Weyl group to unipotent classes, II. arXiv:1104.0196.
[10] G. Lusztig and N. Spaltenstein, On the generalized Springer correspondence for classical groups. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 289-316, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.
[11] N. Spaltenstein, Classes unipotentes et sous-groupes de Borel. (French) Lecture Notes in Mathematics, 946. Springer-Verlag, Berlin-New York, 1982.
[12] N. Spaltenstein, Nilpotent classes and sheets of Lie algebras in bad characteristic. Math. Z. 181 (1982), no. 1, 31-48.
[13] T.A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups. Invent. Math. 36 (1976), 173-207.
[14] T. Xue, Nilpotent orbits in classical Lie algebras over finite fields of characteristic 2 and the Springer correspondence. Represent. Theory 13 (2009), 371-390 (electronic).
[15] T. Xue, Combinatorics of the Springer correspondence for classical Lie algebras and their duals in characteristic 2. arxiv.0911.1350.

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